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Abstract. Domain walls, arising from the spontaneous breaking of a discrete symmetry, can be coupled to charge carriers. In much the same way as the Witten model for a superconducting cosmic string, an investigation is made here into the case of $U(1)\times Z_2 \to U(1)$, where a bosonic charge carrier is directly coupled to the wall-forming Higgs field. All internal quantities, such as the energy per unit surface and the surface current, are calculated numerically to provide the first complete analysis of the internal structure of a surface current-carrying domain wall.

1. Introduction

Domain walls [1, 2] can arise in various grand unified theories (GUT) whenever a discrete symmetry is broken by means of a Higgs field. Because they have immediately been shown to induce a cosmological catastrophe [1] even if they appear in a very late phase transition, their internal structure has not yet been studied in much detail, since it was widely believed that they could not have survived until now. Indeed, with an energy per unit surface of the order of the cube of the symmetry-breaking energy scale η say, a single wall crossing the universe, even with η as low as $\eta \sim 100$ GeV, would produce an enormous overdensity $\Omega_{\rm wall} \sim 10^8$, or, in the case where only small balls were to survive, very large anisotropies in the cosmic microwave background radiation would be induced which are not observed [2]. Hence, if stable walls are to exist in a theory, one must have an inflationary period between the time they were formed and now.

The general belief nowadays concerning domain walls, assuming they were ever produced at all, is examplified by the Peccei–Quinn phase transition [3], whose cosmological relevance notably for the axion problem is still the subject of open discussion [4]. The idea is that even though walls could have been formed, the corresponding phase transition would have been preceded by a string forming transition in such a way that the domain walls would be bounded by strings [5]. In such a framework, all walls would have had a finite size, huge surface tension, and would have evaporated in less than a Hubble time, thereby effectively solving the problem. It could therefore appear that studying their internal structure is indeed pointless.

However, just as in the case of cosmic strings, the situation could be rather different if domain walls were to have the ability to carry some sort of charge. The first immediate effect that has to be considered is that a Bose condensate, in order to be stable in the wall,

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must lower the total energy density. It has been found that this is indeed the case [6], and in fact the energy per unit area in the model we are about to investigate can be made arbitrarily small for some values of the microphysical parameters. This allows a way to accommodate domain walls and cosmology without requiring an inflation period and it should thus be sufficient to justify a deeper analysis of domain walls' internal structure.

Another effect of the currents is similar to what happens in cosmic strings. In this case, a current induces a breaking of the Lorentz symmetry along the worldsheet, so that one can consider rotating loops (called vortons [7] because of their particle-like properties, or rings [8]). The point is that cosmic strings are believed to scale (see [2] for a recent review) because the network of string is dominated by the loops, which eventually gravitationally radiate all their energy away. When a current is present, these loops might reach equilibrium configurations [7, 8] whose classical stability was recently discussed [9, 10] with the result that if no quantum instability exists, the scaling is spoiled and they could easily overfill the universe unless they were produced at a very low energy scale (estimated at $\sim$10 TeV).

Now if the strings bounding the walls were superconducting, the problem could in fact be rather similar, the presence of a domain wall modifying the equilibrium configuration in an unknown way, while presumably modifying the constraint. This issue, which can and should be analysed in the framework of Carter's formalism [11] for describing p-branes, is still a completely open subject. Another difficulty can arise in the case where strings are not current-carrying, but the wall itself is. Indeed, the point is, as before for the cosmic string scenario, that the breaking of the Lorentz symmetry along the worldsheet, whatever its intrinsic dimension, allows a definition of rotation, and eventually the recognition of the existence of centrifugally supported states. Of course, it is not yet clear whether these objects could be formed and reach stable states at all, and therefore their cosmological relevance has not been established. However, in the purpose of studying these frisbee-like configurations, it is necessary that one knows the relevant internal quantities such as the energy per unit area and the surface tensions: they are explicitly calculated in the present article.

Another point to be noted is that domain walls usually appear at the phase transition in the form of closed surfaces that would be expected in a standard nonconducting case to decay very rapidly because of the surface tension. However again, as with strings, the presence of currents can change this situation drastically. Indeed, during the collapse, the charge trapped on the surface can redistribute itself in such a way as to provide an effectively repulsive interaction between the different parts of the wall, thereby opening the possibility that equilibrium configurations appear, with obviously the same problem as in the string–vorton case. Moreover, contrary to the string case, if these last configurations turned out to be unstable and likely to collapse, then one might end up with non-topological soliton configurations [12] which would be formed with a very high charge and therefore would not evaporate [13] and could thus yield a cosmological catastrophe [14]. This is at present an open question.

It may seem that coupling charged (or hypercharged) particles to a domain wall forming Higgs fields is a bit arbitrary, but in view of the fact that most topological defects are predicted to form in various GUT models where the number of degrees of freedom, including scalar, vector and fermion fields is huge, and where the couplings are almost unrestricted, it seems fairly plausible. The purpose of this article is thus to present a toy model, similar to the Witten bosonic model [15] for superconducting cosmic strings, where the symmetry breaking scheme is simply $U(1) \times Z_2 \to U(1)$. This model, much like the Witten model, is expected to yield qualitatively relevant results. The work is arranged as follows. After presenting the actual model in the first section, we investigate the microscopic structure of such a wall and end up by dealing with the above-mentioned integrated internal quantities, namely the energy per unit area, the surface tensions as well as the surface current. The equation of state, relating these quantities, is then computed numerically from the solution of the field equations and is shown to share most of the superconducting cosmic string equation of state properties [16], and in particular the existence of a phase frequency threshold, which is discussed in some length at the end of the paper. This study of a domain wall model thus sheds new light on the general knowledge of current-carrying topological defects by showing, for instance, that a generalization of the string properties in an arbitrary number of dimensions is possible, which in turn gives a new understanding of these string features. With this idea in mind, we end this article by a derivation of the divergent behaviour of the timelike component of the current as a function of the topological defect internal dimension, thus generalizing previous results [16, 17] to arbitrarily dimensioned topological defects.

2. Wall model

Domain walls form whenever a discrete symmetry is spontaneously broken. The simplest way to achieve that is to break a Z_2 symmetry by means of a scalar ϕ whose vacuum expectation value shall be taken as $\langle 0|\varphi|0\rangle = \pm\eta$, with η the energy scale of symmetry breaking. This Higgs field may be coupled with hypercharge-carrying fields which we approximate [15] by a single complex scalar field Σ whose vacuum dynamics we require to be invariant under some $U(1)$ phase transformation group. This $U(1) - \Sigma$ field is not to be mistaken with any string forming field: even though one might be interested in frisbee configurations later on, with strings bounding the wall currently under investigation, it must be emphasized that in general the coupling terms between the wall and the string fields will be of a different kind than that studied here. In much the same way as was done for current-carrying cosmic strings [16, 17], we neglect any long range interaction and thus assume a global $U(1)$ symmetry [16], thereby emphasizing the actual dynamics of the wall, assuming charge-coupling corrections to be negligible, as was shown to be the case for superconducting cosmic strings [17]. The Lagrangian we shall start with is therefore

$$\mathcal{L} = -\tfrac{1}{2}|\partial_\mu\varphi|^2 - \tfrac{1}{2}|\partial_\mu\Sigma|^2 - V(\varphi, \Sigma) \tag{1}$$

with the general interaction potential given by

$$V(\varphi, \Sigma) = \frac{\lambda_\phi}{8}(\varphi^2 - \eta^2)^2 + f|\Sigma|^2(\varphi^2 - \eta^2) + \frac{m_\sigma^2}{2}|\Sigma|^2 + \frac{\lambda_\sigma}{4}|\Sigma|^4. \tag{2}$$

The dynamics given by this Lagrangian includes existence of domain walls, i.e. solutions of the field's equations that separate domains where $\langle 0|\varphi|0\rangle = +\eta$ from regions where $\langle 0|\varphi|0\rangle = -\eta$, and on which therefore $\langle 0|\varphi|0\rangle = 0$. From now on, we shall simply write φ for $\langle 0|\varphi|0\rangle$. The wall solution will be a stationary solution, with the wall locally identified with the (x, y) plane, the various field amplitudes depending only on the third z-coordinate. Note that this choice of symmetry is consistent whenever the characteristic wall curvature is negligible compared with the thickness, a hypothesis whose validity would cease to be true for a non-topological soliton [12].

Our ansatz is thus

$$\varphi = \varphi(z) \qquad \text{and} \qquad \Sigma = \sigma(z)\exp[\mathrm{i}(kx - \omega t)] \tag{3}$$

where we have chosen the frame where the spacelike component of the current, defined below, is directed along the x direction (this form (3) for Σ can always be attained locally by means of a simple rotation in the wall plane). The conserved current, derived as the Noether invariant under phase transformations, is

$$J_\mu = \frac{i}{2}\frac{\delta\mathcal{L}}{\delta\partial^\mu\Sigma}\Sigma^* + \text{c.c.} = \frac{i}{2}\Sigma\overset{\leftrightarrow}{\partial_\mu}\Sigma^* \tag{4}$$

which, with (3) plugged in yields

$$J_\mu = (k\delta_{\mu x} - \omega\delta_{\mu t})\sigma^2(z). \tag{5}$$

The field equations derived from the Lagrangian (1) under the assumptions (3) read

$$\frac{d^2\varphi}{dz^2} = \left[\frac{\lambda_\phi}{2}(\varphi^2 - \eta^2) + 2f\sigma^2\right]\varphi \tag{6}$$

$$\frac{d^2\sigma}{dz^2} = [w + 2f(\varphi^2 - \eta^2) + m_\sigma^2 + \lambda_\sigma\sigma^2]\sigma \tag{7}$$

in which we have defined the state parameter

$$w \equiv k^2 - \omega^2 \tag{8}$$

whose sign reflects the spacelike or timelike character of the current given above by (5) since

$$J_\mu J^\mu = w\sigma^4(z) \tag{9}$$

and in the chosen conventions of (1), the Minkowski metric is $\eta^{\mu\nu} = \text{Diag}\{-1, 1, 1, 1\}$.

The possibility of a current in the wall can be seen in two ways. First, one can notice that the minimum of the potential, in the actual vacuum, is given by

$$\varphi = \pm\eta \qquad \text{and} \qquad \Sigma = 0 \tag{10}$$

and that this minimum is shifted in the wall where $\varphi = 0$ to

$$\lambda_\sigma|\Sigma|^2 = 2f\eta^2 - m_\sigma^2 \tag{11}$$

so a condensate may exist provided

$$m_\sigma^2 \leqslant 2f\eta^2. \tag{12}$$

Another way to realize that a condensate will in fact appear [15] in the wall consists in assuming no condensate ($\Sigma = 0$), and solving the perturbative equation for Σ in the domain wall background. For $\Sigma = 0$, the solution of (6) is known:

$$\varphi = \eta\tanh\left(\tfrac{1}{2}\sqrt{\lambda_\phi}\eta z\right) \tag{13}$$

and setting a perturbation in the form $\Sigma = \sigma(z)e^{i\omega t}$ into (7) yields the one-dimensional Shrödinger equation for σ

$$-\frac{d^2\sigma}{dz^2} + \mathcal{V}(z)\sigma = \omega^2\sigma \tag{14}$$

where the potential

$$\mathcal{V}(z) \equiv -2f\eta^2[1 - \tanh^2(\tfrac{1}{2}\sqrt{\lambda_\phi}\eta z)] + m_\sigma^2 \tag{15}$$

is negative-definite when the condition (12) holds. Hence, under this condition, Σ evolves in an attractive potential well, with negative eigenvalues for ω^2. Therefore, there exists unstable modes and a condensate forms.

3. Current quenching and phase frequency threshold

In order to analyse the internal structure of such a current-carrying domain wall, it turns out to be convenient to introduce a set of dimensionless functions and variables ζ, X, Y, $\tilde{w}$ and $\{\alpha_{1,2,3}\}$ as

$$\varphi(z) = \eta X(\zeta) \tag{16}$$

$$\sigma(z) = \frac{m_\sigma}{\sqrt{\lambda_\sigma}} Y(\zeta) \tag{17}$$

with

$$\zeta = \sqrt{\lambda_\phi}\eta z. \tag{18}$$

The state parameter is similarly rescaled into

$$w = \frac{\lambda_\phi \lambda_\sigma \eta^4}{m_\sigma^2} \tilde{w} \tag{19}$$

and provided we redefine the arbitrary underlying parameters as [16, 17]

$$\alpha_1 = \frac{m_\sigma^2}{\lambda_\sigma \eta^2} \qquad \alpha_2 = \frac{f m_\sigma^2}{\lambda_\phi \lambda_\sigma \eta^2} \qquad \text{and} \qquad \alpha_3 = \frac{m_\sigma^4}{\lambda_\phi \lambda_\sigma \eta^4} \tag{20}$$

we get the very simple set of ordinary differential equations

$$X'' = X[\tfrac{1}{2}(X^2 - 1) + 2\alpha_2 Y^2] \tag{21}$$

$$\alpha_1 Y'' = Y[\tilde{w} + 2\alpha_2(X^2 - 1) + \alpha_3(Y^2 + 1)] \tag{22}$$

where a prime denotes a derivative with respect to ζ.

Two constraints on these parameters arise from the requirement that the theory be physically meaningful and consistent with currents flowing along the wall. The condition (12) for instance, reads in terms of these parameters

$$\alpha_3 \leqslant 2\alpha_2 \tag{23}$$

while demanding that the energy of the wall configuration ($\varphi = 0$ and $\Sigma \neq 0$) be greater than the actual surrounding vacuum configuration ($\varphi = \eta$ and $\Sigma = 0$) implies

$$(\alpha_3 - 2\alpha_2)^2 \leqslant \tfrac{1}{2}\alpha_3. \tag{24}$$

The first of these constraints in fact means that there exists a spacelike saturation current which cannot be exceeded. To see that this is indeed the case, let us perform an expansion of X and Y close to the wall where $\zeta \ll 1$, in the form [16]

$$X \sim x_1 \zeta + b\zeta^3 \qquad \text{and} \qquad Y \sim y_0 - a\zeta^2 \tag{25}$$

which satisfy the boundary conditions on the wall worldsheet, and in particular regularity of the Σ field [which accounts for $Y'(0) = 0$]. Plugging back into (21) and (22) gives

$$b = x_1(2\alpha_2 y_0^2 - \tfrac{1}{2}) \tag{26}$$

and

$$a = \frac{y_0}{2\alpha_2}[2\alpha_2 - \tilde{w} - \alpha_3(y_0^2 + 1)], \tag{27}$$

so that because the condensate is actually at its maximum at $z = 0$, one has $a \geqslant 0$, which means

$$\tilde{w} \leqslant 2\alpha_2 - \alpha_3. \tag{28}$$

Thanks to the requirement (23), we see that the limit applies only in the spacelike current case where $\tilde{w} \geqslant 0$, and it reflects the fact that for $\tilde{w} = 2\alpha_2 - \alpha_3$, one has $y_0 = 0$, and therefore no condensate, hence no current. So there exists a value of the state parameter above which the current quenches to zero.

On the other hand, investigating the large-ζ behaviour of equations (21) and (22) yields the following asymptotics:

$$1 - X \sim \exp(-\zeta) \tag{29}$$

as expected from the knowledge of the true solution (13) in the decoupled case ($X_{\alpha_2=0} \sim 1 - 2e^{\zeta}$), and

$$Y \sim \exp\left(-\sqrt{\frac{\tilde{w}+\alpha_3}{\alpha_1}}\zeta\right) \tag{30}$$

for positive $\tilde{w} + \alpha_3$,

$$Y \sim \cos\left(\sqrt{\left|\frac{\tilde{w}+\alpha_3}{\alpha_1}\right|}\zeta + \delta\right) \tag{31}$$

for negative $\tilde{w} + \alpha_3$, with the special $\tilde{w} = -\alpha_3$ case leading to

$$Y \sim \sqrt{\frac{2\alpha_1}{\alpha_3}}\zeta^{-1}. \tag{32}$$

Thus, exactly as in the case of a current carrying cosmic string, there exists a phase frequency threshold given by $\tilde{w} = -\alpha_3$, or $\omega = m_\sigma$, above which the integral of the current (5) from the sheet to infinity diverges. This is therefore not a mechanism depending on the dimension of the topological defect under consideration, and can be interpreted as charge carrier's evaporation from it [16]. This phase frequency threshold is discussed more thoroughly at the end of the following section where integrated quantities are explicitly calculated.

4. Macroscopic quantities

For most of the cosmologically relevant calculations with topological defects, it is convenient to consider them as infinitely thin, and for that purpose, it is necessary to know the stress–energy tensor and the current as line integrals starting from the wall's worldsheet to infinity. For instance, the integrated current reads

$$\begin{aligned} \mathcal{C} &\equiv 2\int \mathrm{d}z\sqrt{|J_\mu J^\mu|} = 2\sqrt{|w|}\int \mathrm{d}z\,\sigma^2(z) \\ &= 2\eta^2\sqrt{\alpha_1}|\tilde{\nu}|\int \mathrm{d}\zeta\, Y^2(\zeta) \end{aligned} \tag{33}$$

where we have defined $\nu = \mathrm{Sign}(w)\sqrt{|w|}$ and rescaled it according to equation (19); the additional factor of 2 is here to account for both sides of the wall. The parameter ν, being essentially identifiable as k or $-\omega$, is readily interpreted and has thus been used as the relevant parameter for the plots presented below.

Another obviously very useful quantity for a macroscopic description of a surface-current-carrying domain wall is its stress–energy tensor

$$T^{\mu\nu} = -2g^{\mu\alpha}g^{\nu\beta}\frac{\delta\mathcal{L}}{\delta g^{\alpha\beta}} + g^{\mu\nu}\mathcal{L} \tag{34}$$

which, in the case under consideration, needs to be diagonalized. It is worth noting at this point that even though the existence of a current in the wall does indeed break the Lorentz invariance along the worldsheet, thereby raising the stress-energy tensor's degeneracy, it does so through the introduction of one privileged direction. Hence, just as in the string's case, there can be only two different eigenvalues, namely the energy per unit area U, and the surface tension T. The resulting stress-energy tensor then reads

$$\mathbf{T}_{(<0)} \equiv \begin{pmatrix} U & & & \\ & -T & & \\ & & -T & \\ & & & 0 \end{pmatrix} \tag{35}$$

for a timelike current (for which the spatial isotropy is left unbroken), whereas the spacelike current case similarly yields

$$\mathbf{T}_{(>0)} \equiv \begin{pmatrix} U & & & \\ & -U & & \\ & & -T & \\ & & & 0 \end{pmatrix}. \tag{36}$$

We shall now calculate explicitly these eigenvalues in the specific case (1) under consideration, and for that purpose, we perform a Lorentz boost in the x-direction in such a way that the phase of the current carrier Σ reads kz or $-\omega t$. In this frame, in which we shall remain for now on except when it comes to the lightlike case, one has the energy per unit surface

$$U = 2\int \mathrm{d}z\, T_{tt} = \sqrt{\lambda_\phi}\eta^3 \int \mathrm{d}\zeta\, [X'^2 + \alpha_1 Y'^2 + |\tilde{w}|Y^2 + \tfrac{1}{4}(X^2-1)^2 + 2\alpha_2 Y^2(X^2-1) + \alpha^3 Y^2(\tfrac{1}{2}Y^2+1)] \tag{37}$$

the surface tension parallel to the current

$$T_\parallel = -2\int \mathrm{d}z\, T_{xx} = \sqrt{\lambda_\phi}\eta^3 \int \mathrm{d}\zeta [X'^2 + \alpha_1 Y'^2 - |\tilde{w}|Y^2 + \tfrac{1}{4}(X^2-1)^2 + 2\alpha_2 Y^2(X^2-1) + \alpha^3 Y^2(\tfrac{1}{2}Y^2+1)] \tag{38}$$

the surface tension orthogonal to the current

$$T_\perp = -2\int \mathrm{d}z\, T_{yy} = \sqrt{\lambda_\phi}\eta^3 \int \mathrm{d}\zeta\, [X'^2 + \alpha_1 Y'^2 + \tilde{w}Y^2 + \tfrac{1}{4}(X^2-1)^2 + 2\alpha_2 Y^2(X^2-1) + \alpha^3 Y^2(\tfrac{1}{2}2Y^2+1)] \tag{39}$$

while the last integrated component provides a very useful numerical constraint as we shall see shortly because

$$T_z = -2\int \mathrm{d}z\, T_{zz} = \sqrt{\lambda_\phi}\eta^3 \int \mathrm{d}\zeta\, [-X'^2 - \alpha_1 Y'^2 + \tilde{w}Y^2 + \tfrac{1}{4}(X^2-1)^2 + 2\alpha_2 Y^2(X^2-1) + \alpha^3 Y^2(\tfrac{1}{2}Y^2+1)] \tag{40}$$

should in fact vanish identically. This can be checked almost immediately when no condensate is present since in that case, one has $X_0 = \tanh\zeta/2$, so that $X_0' = -\frac{1}{2}(X^2-1)$ which in turn implies

$$T_z^{(0)} = \sqrt{\lambda_\phi}\eta^3 \int \mathrm{d}\zeta\, [-X'^2 + \tfrac{1}{4}(X^2-1)^2] = 0 \tag{41}$$

while the general case gives, with the ansatz (3)

$$\partial_x T^{xx} = \partial_y T^{yy} = \partial_t T^{tt} = 0$$

and finally, conservation of the stress-energy tensor $\partial_\mu T^{\mu\nu} = 0$ yields

$$\partial_z T^{zz} = 0. \tag{42}$$

But the boundary conditions one must use are such that asymptotically, the fields take their vacuum values, so

$$\lim_{z\to\infty} \sigma = \lim_{z\to\infty} \partial_z\sigma = \lim_{z\to\infty} \partial_z\varphi = \lim_{z\to\infty} (\varphi^2 - \eta^2) = 0$$

so $\lim_{z\to\infty} T^{zz} = 0$ which, with (42) implies $T^{zz} = 0$. Hence, (40) provides a constraint on the fields, namely

$$X'^2 + \alpha_1 Y'^2 = \tilde{w} Y^2 + \tfrac{1}{4}(X^2 - 1)^2 + 2\alpha_2 Y^2 (X^2 - 1) + \alpha_3 Y^2 (\tfrac{1}{2} Y^2 + 1) \tag{43}$$

which is used for numerical purposes since it gives the value of the derivative of X near the origin, i.e. x_1 with the notation of (25), as a function of the Σ field's VEV y_0, with

$$x_1^2 = \tfrac{1}{4} + y_0^2[\alpha_3(1 + 2y_0^2) - 2\alpha_2 - \tilde{w}]. \tag{44}$$

Note first that we recover $x_1^2 = \frac{1}{4}$ in the non-current-carrying case, again in agreement with the corresponding known analytic solution, and second that (43) is not a trivial constraint: as numerical integration reveals, the functional $U[X(\zeta), Y(\zeta)]$ has two extrema depending on the field configuration, one of which corresponds to an unphysical maximum, whereas the second is indeed a minimum satisfying (43). The numerical program developed for solving equations (21) and (22) therefore used the constraint (43) by fixing the parameters at the origin with (44). Two criteria for ensuring the convergence to the actual physical solution were thus considered, namely that the solution should indeed be one and therefore should extremize U, and the vanishing of T_z.

A final consideration permits an evaluation of the accuracy of the numerical results thereby obtained, and it is the final point on the ν line calculated for a spacelike current. This point corresponds to $\tilde{w} = 2\alpha_2 - \alpha_3$ which, according to (27) and the discussion following it, has no current at all. In that case, all the integrals of equations (37)–(39) are equal to U_0, with

$$\begin{aligned} U_0 &= \sqrt{\lambda_\phi}\eta^3 \int \mathrm{d}\zeta\, [X_0'^2 + \tfrac{1}{4}(X_0^2 - 1)] \\ &= 2\sqrt{\lambda_\phi}\eta^3 \int \mathrm{d}\zeta\, X_0'^2 \end{aligned} \tag{45}$$

when one takes the solution $X_0 = \tanh \zeta/2$, and this is

$$\begin{aligned} U_0 &= 2\sqrt{\lambda_\phi}\eta^3 \int X'\, \mathrm{d}X = -\sqrt{\lambda_\phi}\eta^3 \int_0^1 (X^2 - 1)\, \mathrm{d}X \\ &= \tfrac{2}{3}\sqrt{\lambda_\phi}\eta^3. \end{aligned} \tag{46}$$

The condensate must therefore respect

$$\frac{U_\sigma}{\sqrt{\lambda_\phi}\eta^3} \leqslant \frac{2}{3} \tag{47}$$

in order to be stable against charge carrier evaporation, with the equality obtained in the limit $\tilde{w} \to 2\alpha_2 - \alpha_3$. This in fact also limits the range of variation of w for a timelike current for it seems doubtful that a state having $U_\sigma > U_0$ could survive in practice.

The case of a lightlike current shares with the non-current-carrying wall the property that the stress–energy tensor's eigenvalues are strictly equal. It can usually be set, after diagonalization for $J_\mu J^\mu \neq 0$, as

$$T^{\mu\nu} = U u^\mu u^\nu - T_\parallel x^\mu x^\nu - T_\perp y^\mu y^\nu \tag{48}$$

with u^μ the timelike eigenvector ($u_\mu u^\mu = -1$) and x^μ, y^μ the spacelike eigenvectors ($x_\mu x^\mu = y_\mu y^\mu = 1$, and $x_\mu y^\mu = 0$), but for a lightlike current, it reads

$$T^{\mu\nu} = U u^\mu u^\nu - T_\| x^\mu x^\nu - T_\perp y^\mu y^\nu - \tfrac{1}{2}(u^\mu x^\nu + u^\nu x^\mu)\omega^2 \int \mathrm{d}z\, \sigma^2(z) \qquad (49)$$

where we have set $\Sigma = \sigma(z)\exp[\mathrm{i}\omega(t-x)]$. Upon diagonalization, this reads

$$T^{\mu\nu} = T_\| (v_-^\mu v_-^\nu - v_+^\mu v_+^\nu - y^\mu y^\nu) \qquad (50)$$

with $v_\pm^\mu = \frac{1}{2}(x^\mu \pm u^\mu)$ the lightlike eigenvectors of $T^{\mu\nu}$, and $T_\|$ as given by (39) with $\tilde{w} = 0$.

Let us investigate more thoroughly the spacelike and timelike cases. The timelike case is characterized, as exemplified in (35), by the isotropy of the purely spatial part of $T^{\mu\nu}$, i.e. $T_\| = T_\perp \equiv T$. As in the string case, one has the Legendre-like relation

$$U - T = -\nu\mathcal{C} \qquad \nu \leqslant 0 \qquad (51)$$

and the now standard formalism developed by Carter [11] applies straightforwardly. The case of a spacelike current is slightly more involved and perhaps requires more thought for each particular cosmologically interesting configuration studied because the spatial isotropy of the surface is no longer present since the current picks a privileged spatial direction in the worldsheet. However, equations (36), (37) and (39) show that yet another simplification arises from the fact that $U = T_\perp$, i.e. the purely spatial component of the stress-energy tensor in the direction parallel to the current flow is the energy per unit surface. Setting $T = T_\|$, a relation similar to (51) is obtained in the form

$$U - T = \nu\mathcal{C} \qquad \nu \geqslant 0 \qquad (52)$$

which can be understood in terms of duality between spacelike and timelike currents [11]. The relevant rescaled integrals are displayed in the figures.

Figure 1 represents the energy per unit area and the surface tensions as functions of the rescaled state parameter $\tilde{\nu}$ for a specific set of parameters $\{\alpha_i\}$ (chosen to yield a generic kind of result as well as giving measurable effects), with curve (a) showing the variations of $U(\tilde{\nu})$ and $T(\tilde{\nu})$ for a spacelike current-carrying wall having a positive state parameter $\tilde{\nu} > 0$, while curve (b) represents $U(\tilde{\nu})$ and $T(\tilde{\nu})$ for a timelike current-carrying wall with $\tilde{\nu} > 0$. Similarly, in figure 2, curves (a) and (b) show the amplitude of the current (5) in the magnetic and electric regimes, respectively. As might have been anticipated, these figures are very much like those obtained for a neutral current-carrying cosmic string [16], at least in the classically stable part of the equation of state, which is definable through the requirement that the soundlike perturbation squared velocity $c_L^2 = -\mathrm{d}T/\mathrm{d}U$ be positive. Thus, the approximate analytic equation of state proposed in [18] should also be useful in this domain wall context. Therefore, most of the current-carrying domain wall properties are essentially similar to the string properties.

Finally, let us remark on the following important mathematical property of the surface current-carrying domain wall. As is the case for a superconducting cosmic string, it can be seen that there exists a phase frequency threshold [16] given by $w = -m_\sigma^2$ at which point the current (33) diverges. For the cosmic string case, the first-order pole behaviour $\mathcal{C}_{\text{string}} \sim (w + m_\sigma^2)^{-1}$ was found [16] whereas the wall case yields $\mathcal{C}_{\text{wall}} \sim (w + m_\sigma^2)^{-1/2}$. This is because in both cases, denoting by d the co-dimension of the topological defect, i.e. 2 for a string and 1 for a wall in a four-dimensional background, the current carrier field is seen to satisfy (7) which, far from the topological defect, gives the relation

$$\Delta_d \sigma \sim (w + m_\sigma^2)\sigma \qquad (53)$$

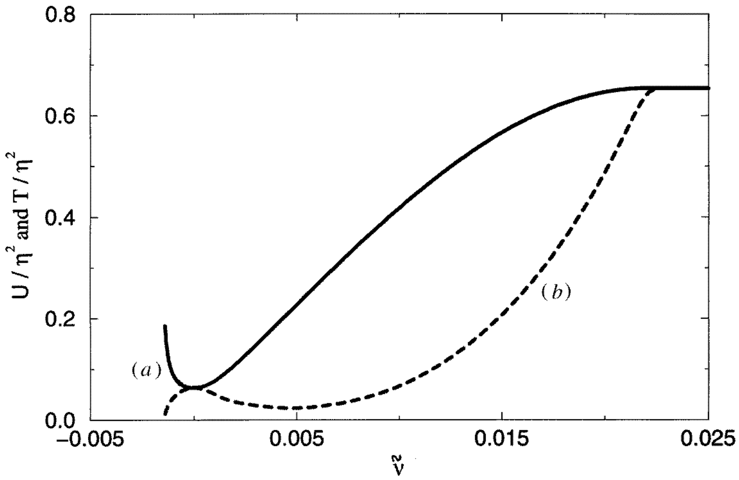

Figure 1. Energy per unit area U (full curve) and surface tensions $T_{>0} = T_{\parallel}$ and $T_{<0} = T_{\perp}$ (broken curves) as functions of the rescaled state parameter $\tilde{\nu}$ and in units of $\sqrt{\lambda_\phi}\eta^3$; ($a$) is for the timelike case $\tilde{\nu} \leqslant 0$, while (b) is for the spacelike range $\tilde{\nu} \geqslant 0$. The calculation is made for a quite extreme set of parameters for which the minimum energy (for $\tilde{\nu} = 0$) is much smaller than the non-current-carrying configuration energy [6]. It should be noted that even for this extreme set of parameters, the tension remains positive in the range (b), providing a hint that the 'no-spring conjecture' [17] may apply independently of the defect dimension. In the timelike current regime, the calculation is stopped when the tension becomes negative in which case the wall is unstable [9].

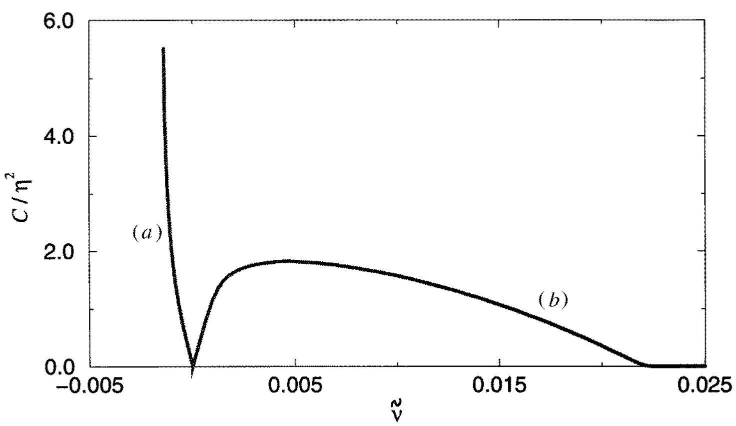

Figure 2. Integrated value of the surface current in units of η^2 for the same variation ranges as in figure 1 as a function of $\tilde{\nu}$. The phase frequency threshold divergence in the electric regime (a) is more visible than in figure 1, and the current saturation phenomenon for the magnetic regime (b) that exists in string models of the same kind [16] is seen to be present as well in the wall case.

where Δ_d stands for the Laplacian in d dimensions: this is simply $\mathrm{d}^2/\mathrm{d}z^2$ in the wall case under consideration here, and $\mathrm{d}^2/\mathrm{d}r^2 + \frac{d-1}{r}\,\mathrm{d}/\mathrm{d}r$ in the general case with r the 'radial' distance to the defect's core. Setting $\chi = kr$, with $r \equiv z$ in our wall case and $k^2 = w + m_\sigma^2$, one can extract σ as a function of k since for $k \neq 0$, equation (7) (i.e. equation (53)) transforms into

$$\frac{d^2\sigma}{d\chi^2} + \frac{d-1}{\chi}\frac{d\sigma}{d\chi} = \sigma(\chi) \tag{54}$$

whose solution cannot depend on k.

The solution to (54) is well known:

$$\sigma \sim A\chi^{1-d/2}K_0(\chi) \tag{55}$$

with K_0 the Bessel function of zeroth order whose asymptotic behaviour is given by $K_0(\chi) \sim \exp(-\chi)/\sqrt{\chi}$. Thus, one finds the general phase frequency threshold behaviour, up to a finite part (corresponding to the fact that one has to integrate up to the point where the approximation (53) becomes valid)

$$\begin{aligned} \mathcal{C} &\propto \int r^{d-1}\,dr\,\sigma^2(kr) \\ &\propto \frac{1}{k^d}\int \chi K_0^2(\chi)\,d\chi \\ &\propto (w + m_\sigma^2)^{-d/2} \end{aligned} \tag{56}$$

with $d = 1$ for a current-carrying domain wall, $d = 2$ for a superconducting cosmic string, and $d = 3$ for a charged monopole in a four-dimensional background spacetime. It is in fact possible to be slightly more precise concerning this divergence: the function κ, defined as [11, 15]

$$\kappa \equiv 2\frac{dU}{dw} = 2\int d^d\boldsymbol{x}_\perp \sigma^2(\boldsymbol{x}_\perp) \tag{57}$$

being proportional to $\mathcal{C}$, also diverges, and it may be seen that, under the assumption that $y_0^2 \sim 2\alpha_2/\alpha_3$ near the threshold (see [16] and (27))

$$\kappa = \kappa_f(w) + A\frac{f\eta^2}{\lambda_\sigma}(w + m_\sigma^2)^{-d/2} \tag{58}$$

which is valid for various values of the co-dimension d, with $\kappa_f(w)$ the finite part of κ and A a pure number, calculable in principle by a matching of the asymptotic solution (55) to the origin and depending on d. Note that the dimension of this function κ is given straightforwardly once d is known.

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